The n-th root of sequential effect algebras*

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Abstract

In 2005, Professor Gudder presented 25 open problems of sequential effect algebras, the 20th problem asked: In a sequential effect algebra, if the square root of some element exists, is it unique ? We can strengthen the problem as following: For each given positive integer $n > 1$, is there a sequential effect algebra such that the n-th root of its some element c is not unique and the n-th root of c is not the k-th root of c ($k < n$) ? In this paper, we answer the strengthened problem affirmatively.

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Let H be a complex Hilbert space and $\mathcal{D}(H)$ the set of density operators on H, i.e., the trace class positive operators on H of unit trace, which represent the states of quantum system. A self-adjoint operator A on H such that $0 \le A \le I$ is called a *quantum effect* ([1, 2]), the set of quantum effects on H is denoted by $\mathcal{E}(H)$. The set of orthogonal projection operators on H is denoted by $\mathcal{P}(H)$. For each $P \in \mathcal{P}(H)$ is associated a so-called Lüders transformation $\Phi_L^P : \mathcal{D}(H) \to \mathcal{D}(H)$ such that for each $T \in \mathcal{D}(H)$, $\Phi_L^P(T) = PTP$. Moreover, each quantum effect $B \in \mathcal{E}(H)$ gives also to a

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general Lüders transformation Φ_L^B such that for each $T \in \mathcal{D}(H)$, $\Phi_L^B(T) = B^{\frac{1}{2}}TB^{\frac{1}{2}}$ ([3-4]).

Let $B, C \in \mathcal{E}(H)$ be two quantum effects. It is easy to prove that the composition $\Phi_L^B \circ \Phi_L^C$ satisfies that for each $T \in \mathcal{D}(H)$, $(\Phi_L^B \circ \Phi_L^C)(T) = (B^{\frac{1}{2}}CB^{\frac{1}{2}})^{\frac{1}{2}}T(B^{\frac{1}{2}}CB^{\frac{1}{2}})^{\frac{1}{2}}$ ([4]). Professor Gudder called $B^{\frac{1}{2}}CB^{\frac{1}{2}}$ the *sequential product* of B and C, and denoted it by $B \circ C$ ([5-7]). This sequential product has been generalized to an algebraic structure called a *sequential effect algebra* ([8]). Now, we state the basic definitions and results of sequential effect algebras.

An *effect algebra* is a system $(E, 0, 1, \oplus)$, where 0 and 1 are distinct elements of E and $\oplus$ is a partial binary operation on E satisfying that [9]:

(EA1). If $a \oplus b$ is defined, then $b \oplus a$ is defined and $b \oplus a = a \oplus b$.

(EA2). If $a \oplus (b \oplus c)$ is defined, then $(a \oplus b) \oplus c$ is defined and

$$(a \oplus b) \oplus c = a \oplus (b \oplus c).$$

(EA3). For each $a \in E$, there exists a unique element $b \in E$ such that $a \oplus b = 1$.

(EA4). If $a \oplus 1$ is defined, then $a = 0$.

In an effect algebra $(E, 0, 1, \oplus)$, if $a \oplus b$ is defined, we write $a \perp b$. For each $a \in (E, 0, 1, \oplus)$, it follows from (EA3) that there exists a unique element $b \in E$ such that $a \oplus b = 1$, we denote b by a'. Let $a, b \in (E, 0, 1, \oplus)$, if there exists a $c \in E$ such that $a \perp c$ and $a \oplus c = b$, then we say that $a \leq b$, if in addition, $a \neq b$, then we write $a < b$. It follows from [9] that $\leq$ is a partial order of $(E, 0, 1, \oplus)$ and satisfies that for each $a \in E$, $0 \leq a \leq 1$, $a \perp b$ if and only if $a \leq b'$.

A *sequential effect algebra* is an effect algebra $(E, 0, 1, \oplus)$ and another binary operation $\circ$ defined on $(E, 0, 1, \oplus)$ satisfying that [8]:

(SEA1). The map $b \mapsto a \circ b$ is additive for each $a \in E$, that is, if $b \perp c$, then $a \circ b \perp a \circ c$ and $a \circ (b \oplus c) = a \circ b \oplus a \circ c$.

(SEA2). $1 \circ a = a$ for each $a \in E$.

(SEA3). If $a \circ b = 0$, then $a \circ b = b \circ a$.

(SEA4). If $a \circ b = b \circ a$, then $a \circ b' = b' \circ a$ and $a \circ (b \circ c) = (a \circ b) \circ c$ for each $c \in E$.

(SEA5). If $c \circ a = a \circ c$ and $c \circ b = b \circ c$, then $c \circ (a \circ b) = (a \circ b) \circ c$ and $c \circ (a \oplus b) = (a \oplus b) \circ c$ whenever $a \perp b$.

Let $(E, 0, 1, \oplus, \circ)$ be a sequential effect algebra. Then the operation $\circ$ is said to be a *sequential product* on $(E, 0, 1, \oplus, \circ)$. If $a, b \in (E, 0, 1, \oplus, \circ)$ and $a \circ b = b \circ a$, then a and b is said to be *sequentially independent* and write $a|b$ ([8]). Let $a \in (E, 0, 1, \oplus, \circ)$. If there exists an element $b \in (E, 0, 1, \oplus, \circ)$ such that $\underbrace{b \circ b \circ \cdots \circ b}_{the\ number\ is\ n} = a$, then we write $b^n = a$ and b is said to be a *n-th root* of a. Note that b is a n-th root of a implies that a can be obtained by measuring b n-times repeatedly.

The sequential effect algebra is an important and interesting mathematical model for studying the quantum measurement theory [5-8]. In [10], Professor Gudder presented 25 open problems to motivate the study of sequential effect algebra theory. The 20th problem asked:

Problem 1 ([10]). In a sequential effect algebra $(E, 0, 1, \oplus, \circ)$, if the square root of some element exists, is it unique ?

Now, we can strengthen Problem 1 as following:

Problem 2. For each given positive integer $n > 1$, is there a sequential effect algebra $(E, 0, 1, \oplus, \circ)$ such that the n-th root of its some element c is not unique and the n-th root of c is not the k-th root of c $(k < n)$? i.e., are there $a, b \in E$, such that $a \neq b$, $a^n = c = b^n$ and $a^k \neq c$, $b^k \neq c$ for $k < n$?

In this paper, we present an example to answer Problem 2 affirmatively. Actually, we will construct a sequential effect algebra E_0, such that there are elements $a, b, c \in E_0$ having the relations

$$a > a^2 > \cdots > a^n,$$

$$b > b^2 > \cdots > b^n,$$

$$a^k \neq b^k \ for \ k < n \ , \ a^n = b^n = c \neq 0.$$

In order to construct our example, we need some preliminary steps:

Suppose Z be the integer set, $n > 1$ be a given positive integer.

Let $p(x)=\sum\limits_{i=1}^{n-1} k_i x^i$, where $k_i \in Z$, $k_i \equiv 0$ or the first nonzero $k_i > 0$, we denote all the polynomials characterized above by I_0 .

Suppose $p_1, p_2 \in I_0$ and $p_1(x)=\sum\limits_{i=1}^{n-1} k_{1,i}x^i$, $p_2(x)=\sum\limits_{i=1}^{n-1} k_{2,i}x^i$, let $F(p_1,p_2)(x)=\sum\limits_{i+j\leq n-1} k_{1,i}k_{2,j}x^{i+j}$, $G(p_1,p_2)=\sum\limits_{i+j=n} k_{1,i}k_{2,j}$. Then it is easy to see that $F(p_1,p_2)\in I_0$ and $G(p_1,p_2)\in Z$.

Thus we defined mappings

$$F : I_0 \times I_0 \longrightarrow I_0 \quad \text{and} \quad G : I_0 \times I_0 \longrightarrow Z \ .$$

Moreover, suppose $p_1, p_2, p_3 \in I_0$ and $p_1(x)=\sum\limits_{i=1}^{n-1} k_{1,i}x^i$, $p_2(x)=\sum\limits_{i=1}^{n-1} k_{2,i}x^i$, $p_3(x)=\sum\limits_{i=1}^{n-1} k_{3,i}x^i$, let $\overline{F}(p_1,p_2,p_3)(x)=\sum\limits_{i+j+m\leq n-1} k_{1,i}k_{2,j}k_{3,m}x^{i+j+m}$, $\overline{G}(p_1,p_2,p_3)=\sum\limits_{i+j+m=n} k_{1,i}k_{2,j}k_{3,m}$. Then it is also easy to see that $\overline{F}(p_1,p_2,p_3)\in I_0$ and $\overline{G}(p_1,p_2,p_3)\in Z$. Thus we defined mappings

$$\overline{F} : I_0 \times I_0 \times I_0 \longrightarrow I_0 \quad \text{and} \quad \overline{G} : I_0 \times I_0 \times I_0 \longrightarrow Z \ .$$

Lemma 1. Suppose $p, p_1, p_2, p_3 \in I_0$, we have

(1). $F(p_1,p_2)=F(p_2,p_1)$, $G(p_1,p_2)=G(p_2,p_1)$;

(2). $F(p_1,p_2+p_3)=F(p_1,p_2)+F(p_1,p_3)$, $G(p_1,p_2+p_3)=G(p_1,p_2)+G(p_1,p_3)$;

(3). $F(0,p)=0$, $G(0,p)=0$;

(4). if $F(p_1,p_2)=0$, then $G(p_1,p_2)\geq 0$;

(5). $p_1-F(p_1,p_2)\in I_0$, and $p_1=F(p_1,p_2)\Longleftrightarrow p_1=0$;

(6). $F(F(p_1,p_2),p_3)=\overline{F}(p_1,p_2,p_3)$, $G(F(p_1,p_2),p_3)=\overline{G}(p_1,p_2,p_3)$;

(7). $p_1+p_2\in I_0$, and $p_1+p_2=0\Longleftrightarrow p_1=p_2=0$.

Proof. (1),(2),(3),(6) and (7) are trivial.

(4). Except for the trivial cases, we may suppose $p_1(x)=\sum\limits_{i=n_1}^{n-1} k_{1,i}x^i$, $p_2(x)=\sum\limits_{i=n_2}^{n-1} k_{2,i}x^i$, with $k_{1,n_1}>0$ and $k_{2,n_2}>0$. Then from $F(p_1,p_2)=0$ we have $n_1+n_2\geq n$. If $n_1+n_2=n$, then $G(p_1,p_2)=k_{1,n_1}k_{2,n_2}>0$; otherwise $n_1+n_2>n$ and $G(p_1,p_2)=0$.

(5). Except for the trivial cases, we may suppose $p_1(x)=\sum\limits_{i=n_1}^{n-1} k_{1,i}x^i$, $p_2(x)=\sum\limits_{i=n_2}^{n-1} k_{2,i}x^i$, with $k_{1,n_1}>0$ and $k_{2,n_2}>0$. Then the first item of $p_1-F(p_1,p_2)$ is

$k_{1,n_1}x^{n_1}$, so $p_1 - F(p_1, p_2) \in I_0$. If $p_1 \neq 0$, then from the above reason we know that $p_1 - F(p_1, p_2) \neq 0$. Thus, the lemma is proved.

Now, we take two infinite sets U and V such that $U \cap V = \emptyset$. Let $f : I_0 \times I_0 \times Z \to U$ and $g : I_0 \times I_0 \times Z \to V$ be two one to one maps. Then, we construct our example as following:

Let $E_0 = \{f(p,q,m), g(p,q,m) | p, q \in I_0, m \in Z \ and \ satisfy \ that \ m \geq 0 \ whenever \ p = q = 0\}$.

First, we define a partial binary operation $\oplus$ on E_0 as follows (when we write $x \oplus y = z$, we always mean that $x \oplus y = z = y \oplus x$):

(i). $f(p_1, q_1, m_1) \oplus f(p_2, q_2, m_2) = f(p_1 + p_2, q_1 + q_2, m_1 + m_2)$ (the right side is well-defined, see Lemma 1(7));

(ii). for $p_2 - p_1 \in I_0$, $q_2 - q_1 \in I_0$, and satisfy that $m_2 \geq m_1$ when $p_2 = p_1$ and $q_2 = q_1$, $f(p_1, q_1, m_1) \oplus g(p_2, q_2, m_2) = g(p_2 - p_1, q_2 - q_1, m_2 - m_1)$.

No other $\oplus$ operation is defined.

Next, we define a binary operation $\circ$ on E_0 as follows (when we write $x \circ y = z$, we always mean that $x \circ y = z = y \circ x$):

(i). $f(p_1, q_1, m_1) \circ f(p_2, q_2, m_2) = f\Big(F(p_1, p_2), F(q_1, q_2), G(p_1, p_2) + G(q_1, q_2)\Big)$ (the right side is well-defined, see Lemma 1(4));

(ii). $f(p_1, q_1, m_1) \circ g(p_2, q_2, m_2) = f\Big(p_1 - F(p_1, p_2), q_1 - F(q_1, q_2), m_1 - G(p_1, p_2) - G(q_1, q_2)\Big)$ (the right side is well-defined, see Lemma 1(3), (5));

(iii). $g(p_1, q_1, m_1) \circ g(p_2, q_2, m_2) = g\Big(p_1 + p_2 - F(p_1, p_2), q_1 + q_2 - F(q_1, q_2), m_1 + m_2 - G(p_1, p_2) - G(q_1, q_2)\Big)$ (the right side is well-defined, see Lemma 1(3), (5), (7)).

We denote $f(0,0,0)$ by 0, $g(0,0,0)$ by 1.

Proposition 1. $(E_0, 0, 1, \oplus, \circ)$ *is a sequential effect algebra.*

Proof. In the proof below, we will use Lemma 1 frequently without annotation. First, we verify that $(E_0, 0, 1, \oplus)$ is an effect algebra.

(EA1) is obvious. We verify (EA2) as follows:

(i). $f(p_1,q_1,m_1)\oplus\Big(f(p_2,q_2,m_2)\oplus f(p_3,q_3,m_3)\Big)=\Big(f(p_1,q_1,m_1)\oplus f(p_2,q_2,m_2)\Big)\oplus f(p_3,q_3,m_3)=f(p_1+p_2+p_3,q_1+q_2+q_3,m_1+m_2+m_3)$;

(ii). $f(p_1,q_1,m_1)\oplus\Big(f(p_2,q_2,m_2)\oplus g(p_3,q_3,m_3)\Big)$ or $\Big(f(p_1,q_1,m_1)\oplus f(p_2,q_2,m_2)\Big)\oplus g(p_3,q_3,m_3)$ is defined iff $p_3-p_1-p_2\in I_0$, $q_3-q_1-q_2\in I_0$ and satisfy that $m_3\geq m_1+m_2$ when $p_3=p_1+p_2$ and $q_3=q_1+q_2$, at this point, they all equal to $g(p_3-p_1-p_2,q_3-q_1-q_2,m_3-m_1-m_2)$.

Note that $f(p,q,m)\oplus g(p,q,m)=g(0,0,0)=1$, we verified (EA3).

For (EA4), we note from our construction that the unique element orthogonal to $g(0,0,0)(=1)$ is $f(0,0,0)(=0)$, that is, $f(0,0,0)\perp g(0,0,0)$ and $f(0,0,0)\oplus g(0,0,0)=g(0,0,0)$.

So far, we have proved that $(E_0,0,1,\oplus)$ is an effect algebra.

Next, we verify that $(E_0,0,1,\oplus,\circ)$ is a sequential effect algebra.

(SEA3) and (SEA5) are obvious.

We verify (SEA1) as follows:

(i). $f(p_1,q_1,m_1)\circ\Big(f(p_2,q_2,m_2)\oplus f(p_3,q_3,m_3)\Big)=f(p_1,q_1,m_1)\circ f(p_2,q_2,m_2)\oplus f(p_1,q_1,m_1)\circ f(p_3,q_3,m_3)=f\Big(F(p_1,p_2+p_3),F(q_1,q_2+q_3),G(p_1,p_2+p_3)+G(q_1,q_2+q_3)\Big)$,

$g(p_1,q_1,m_1)\circ\Big(f(p_2,q_2,m_2)\oplus f(p_3,q_3,m_3)\Big)=g(p_1,q_1,m_1)\circ f(p_2,q_2,m_2)\oplus g(p_1,q_1,m_1)\circ f(p_3,q_3,m_3)=f\Big(p_2+p_3-F(p_1,p_2+p_3),q_2+q_3-F(q_1,q_2+q_3),m_2+m_3-G(p_1,p_2+p_3)-G(q_1,q_2+q_3)\Big)$;

(ii). when $f(p_2,q_2,m_2)\oplus g(p_3,q_3,m_3)$ is defined, i.e., when $p_3-p_2\in I_0$, $q_3-q_2\in I_0$, and satisfy that $m_3\geq m_2$ if $p_3=p_2$ and $q_3=q_2$,

$f(p_1,q_1,m_1)\circ\Big(f(p_2,q_2,m_2)\oplus g(p_3,q_3,m_3)\Big)=f(p_1,q_1,m_1)\circ f(p_2,q_2,m_2)\oplus f(p_1,q_1,m_1)\circ g(p_3,q_3,m_3)=f\Big(p_1-F(p_1,p_3-p_2),q_1-F(q_1,q_3-q_2),m_1-G(p_1,p_3-p_2)-G(q_1,q_3-q_2)\Big)$,

$g(p_1,q_1,m_1)\circ\Big(f(p_2,q_2,m_2)\oplus g(p_3,q_3,m_3)\Big)=g(p_1,q_1,m_1)\circ f(p_2,q_2,m_2)\oplus g(p_1,q_1,m_1)\circ g(p_3,q_3,m_3)=g\Big(p_1+p_3-p_2-F(p_1,p_3-p_2),q_1+q_3-q_2-F(q_1,q_3-q_2),m_1+m_3-m_2-G(p_1,p_3-p_2)-G(q_1,q_3-q_2)\Big)$.

We verify (SEA2) as follows:

$$1 \circ f(p,q,m) = g(0,0,0) \circ f(p,q,m) = f(p,q,m);$$

$$1 \circ g(p,q,m) = g(0,0,0) \circ g(p,q,m) = g(p,q,m).$$

We verify (SEA4) as follows:

(i). $f(p_1,q_1,m_1) \circ \Big(f(p_2,q_2,m_2) \circ f(p_3,q_3,m_3)\Big)$
$= f(p_1,q_1,m_1) \circ f\Big(F(p_2,p_3), F(q_2,q_3), G(p_2,p_3) + G(q_2,q_3)\Big)$
$= f\Big(F(p_1,F(p_2,p_3)), F(q_1,F(q_2,q_3)), G(p_1,F(p_2,p_3)) + G(q_1,F(q_2,q_3))\Big)$
$= f\Big(\overline{F}(p_1,p_2,p_3), \overline{F}(q_1,q_2,q_3), \overline{G}(p_1,p_2,p_3) + \overline{G}(q_1,q_2,q_3)\Big),$

by symmetry,
$\Big(f(p_1,q_1,m_1) \circ f(p_2,q_2,m_2)\Big) \circ f(p_3,q_3,m_3)$
$= f(p_3,q_3,m_3) \circ \Big(f(p_1,q_1,m_1) \circ f(p_2,q_2,m_2)\Big)$
$= f\Big(\overline{F}(p_1,p_2,p_3), \overline{F}(q_1,q_2,q_3), \overline{G}(p_1,p_2,p_3) + \overline{G}(q_1,q_2,q_3)\Big),$

so we have
$f(p_1,q_1,m_1) \circ \Big(f(p_2,q_2,m_2) \circ f(p_3,q_3,m_3)\Big) = \Big(f(p_1,q_1,m_1) \circ f(p_2,q_2,m_2)\Big) \circ f(p_3,q_3,m_3)$.

(ii). $f(p_1,q_1,m_1) \circ \Big(f(p_2,q_2,m_2) \circ g(p_3,q_3,m_3)\Big)$
$= f(p_1,q_1,m_1) \circ f\Big(p_2 - F(p_2,p_3), q_2 - F(q_2,q_3), m_2 - G(p_2,p_3) - G(q_2,q_3)\Big)$
$= f\Big(F(p_1,p_2 - F(p_2,p_3)), F(q_1,q_2 - F(q_2,q_3)), G(p_1,p_2 - F(p_2,p_3)) + G(q_1,q_2 - F(q_2,q_3))\Big)$
$= f\Big(F(p_1,p_2) - F(p_1,F(p_2,p_3)), F(q_1,q_2) - F(q_1,F(q_2,q_3)), G(p_1,p_2) - G(p_1,F(p_2,p_3)) + G(q_1,q_2) - G(q_1,F(q_2,q_3))\Big)$
$= f\Big(F(p_1,p_2) - \overline{F}(p_1,p_2,p_3), F(q_1,q_2) - \overline{F}(q_1,q_2,q_3), G(p_1,p_2) - \overline{G}(p_1,p_2,p_3) + G(q_1,q_2) - \overline{G}(q_1,q_2,q_3)\Big),$

$\Big(f(p_1,q_1,m_1) \circ f(p_2,q_2,m_2)\Big) \circ g(p_3,q_3,m_3)$
$= f\Big(F(p_1,p_2), F(q_1,q_2), G(p_1,p_2) + G(q_1,q_2)\Big) \circ g(p_3,q_3,m_3)$
$= f\Big(F(p_1,p_2) - F(F(p_1,p_2),p_3), F(q_1,q_2) - F(F(q_1,q_2),q_3), G(p_1,p_2) + G(q_1,q_2) - G(F(p_1,p_2),p_3) - G(F(q_1,q_2),q_3)\Big)$

$$= f\Big(F(p_1,p_2) - \overline{F}(p_1,p_2,p_3), F(q_1,q_2) - \overline{F}(q_1,q_2,q_3), G(p_1,p_2) - \overline{G}(p_1,p_2,p_3) + G(q_1,q_2) - \overline{G}(q_1,q_2,q_3)\Big),$$

so we have

$f(p_1,q_1,m_1) \circ \Big(f(p_2,q_2,m_2) \circ g(p_3,q_3,m_3)\Big) = \Big(f(p_1,q_1,m_1) \circ f(p_2,q_2,m_2)\Big) \circ g(p_3,q_3,m_3)$.

(iii). $f(p_1,q_1,m_1) \circ \Big(g(p_2,q_2,m_2) \circ g(p_3,q_3,m_3)\Big)$

$$= f(p_1,q_1,m_1) \circ g\Big(p_2+p_3-F(p_2,p_3), q_2+q_3-F(q_2,q_3), m_2+m_3-G(p_2,p_3)-G(q_2,q_3)\Big)$$
$$= f\Big(p_1 - F(p_1,p_2+p_3-F(p_2,p_3)), q_1 - F(q_1,q_2+q_3-F(q_2,q_3)), m_1 - G(p_1,p_2+p_3-F(p_2,p_3)) - G(q_1,q_2+q_3-F(q_2,q_3))\Big)$$
$$= f\Big(p_1 - F(p_1,p_2+p_3) + \overline{F}(p_1,p_2,p_3), q_1 - F(q_1,q_2+q_3) + \overline{F}(q_1,q_2,q_3), m_1 - G(p_1,p_2+p_3) + \overline{G}(p_1,p_2,p_3) - G(q_1,q_2+q_3) + \overline{G}(q_1,q_2,q_3)\Big),$$

$\Big(f(p_1,q_1,m_1) \circ g(p_2,q_2,m_2)\Big) \circ g(p_3,q_3,m_3)$

$$= f\Big(p_1 - F(p_1,p_2), q_1 - F(q_1,q_2), m_1 - G(p_1,p_2) - G(q_1,q_2)\Big) \circ g(p_3,q_3,m_3)$$
$$= f\Big(p_1 - F(p_1,p_2) - F(p_1 - F(p_1,p_2),p_3), q_1 - F(q_1,q_2) - F(q_1 - F(q_1,q_2),q_3), m_1 - G(p_1,p_2) - G(q_1,q_2) - G(p_1 - F(p_1,p_2),p_3) - G(q_1 - F(q_1,q_2),q_3)\Big)$$
$$= f\Big(p_1 - F(p_1,p_2+p_3) + \overline{F}(p_1,p_2,p_3), q_1 - F(q_1,q_2+q_3) + \overline{F}(q_1,q_2,q_3), m_1 - G(p_1,p_2+p_3) + \overline{G}(p_1,p_2,p_3) - G(q_1,q_2+q_3) + \overline{G}(q_1,q_2,q_3)\Big),$$

so we have

$f(p_1,q_1,m_1) \circ \Big(g(p_2,q_2,m_2) \circ g(p_3,q_3,m_3)\Big) = \Big(f(p_1,q_1,m_1) \circ g(p_2,q_2,m_2)\Big) \circ g(p_3,q_3,m_3)$.

(iv). $g(p_1,q_1,m_1) \circ \Big(g(p_2,q_2,m_2) \circ g(p_3,q_3,m_3)\Big)$

$$= g(p_1,q_1,m_1) \circ g\Big(p_2+p_3-F(p_2,p_3), q_2+q_3-F(q_2,q_3), m_2+m_3-G(p_2,p_3)-G(q_2,q_3)\Big)$$
$$= g\Big(p_1+p_2+p_3-F(p_2,p_3)-F(p_1,p_2+p_3-F(p_2,p_3)), q_1+q_2+q_3-F(q_2,q_3) - F(q_1,q_2+q_3-F(q_2,q_3)), m_1+m_2+m_3-G(p_2,p_3)-G(q_2,q_3)-G(p_1,p_2+p_3-F(p_2,p_3)) - G(q_1,q_2+q_3-F(q_2,q_3))\Big)$$
$$= g\Big(p_1+p_2+p_3-F(p_2,p_3)-F(p_1,p_2)-F(p_1,p_3)+\overline{F}(p_1,p_2,p_3), q_1+q_2+q_3 - F(q_2,q_3)-F(q_1,q_2)-F(q_1,q_3)+\overline{F}(q_1,q_2,q_3), m_1+m_2+m_3-G(p_2,p_3) - G(p_1,p_2)-G(p_1,p_3)+\overline{G}(p_1,p_2,p_3)-G(q_2,q_3)-G(q_1,q_2)-G(q_1,q_3)+$$

$\overline{G}(q_1, q_2, q_3)\Big)$,

by symmetry, we have

$g(p_1, q_1, m_1) \circ \Big(g(p_2, q_2, m_2) \circ g(p_3, q_3, m_3)\Big) = \Big(g(p_1, q_1, m_1) \circ g(p_2, q_2, m_2)\Big) \circ g(p_3, q_3, m_3)$.

Thus, we proved that $(E_0, 0, 1, \oplus, \circ)$ is a sequential effect algebra and the theorem is proved.

Now, let $P_i(x) = x^i$. Then it is easy to see that

$$F(P_1, P_j) = \begin{cases} P_{1+j}\ , & if\ j < n-1; \\ 0\ , & if\ j = n-1. \end{cases} \quad and \quad G(P_1, P_j) = \begin{cases} 0\ , & if\ j < n-1; \\ 1\ , & if\ j = n-1. \end{cases}$$

Thus we have

$$[f(P_1, 0, 0)]^k = f(P_1, 0, 0) \circ f(P_{k-1}, 0, 0) = f(P_k, 0, 0) \text{ for } k < n,$$

$$[f(P_1, 0, 0)]^n = f(P_1, 0, 0) \circ f(P_{n-1}, 0, 0) = f(0, 0, 1),$$

$$[f(P_1, 0, 0)]^{n+1} = f(P_1, 0, 0) \circ f(0, 0, 1) = 0,$$

and

$$[f(0, P_1, 0)]^k = f(0, P_1, 0) \circ f(0, P_{k-1}, 0) = f(0, P_k, 0) \text{ for } k < n,$$

$$[f(0, P_1, 0)]^n = f(0, P_1, 0) \circ f(0, P_{n-1}, 0) = f(0, 0, 1),$$

$$[f(0, P_1, 0)]^{n+1} = f(0, P_1, 0) \circ f(0, 0, 1) = 0.$$

If we denote $f(P_1, 0, 0)$ by a, $f(0, P_1, 0)$ by b, $f(0, 0, 1)$ by c, then it is easy to get the relations

$$a > a^2 > \cdots > a^n > a^{n+1},$$

$$b > b^2 > \cdots > b^n > b^{n+1},$$

$$a^k \neq b^k\ for\ k < n\ ,\ a^n = b^n = c \neq 0\ and\ a^{n+1} = b^{n+1} = 0.$$

That is, a, b are the n-th root of c, but a, b are not the k-th root of c, where $k = 2, 3, \cdots, n-1$, moreover, a, b are also the n+1-th root of 0, so, the Problem 2 is answered affirmatively.

Finally, we would like to point out that for the advances of sequential effect algebras, see [11-16].

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